

ON FAST IMPLEMENTATION OF HIGHER ORDER HERMITE-FEJÉR INTERPOLATION*

SHUHUANG XIANG[†] AND GUO HE^{‡†}

Abstract. The problem of barycentric Hermite interpolation is highly susceptible to overflows or underflows. In this paper, based on Sturm-Liouville equations for Jacobi orthogonal polynomials, we consider the fast implementation on the second barycentric formula for higher order Hermite-Fejér interpolation at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems, where the barycentric weights can be efficiently evaluated and cost linear operations corresponding to the number of grids totally. Furthermore, due to the division of the second barycentric form, the exponentially increasing common factor in the barycentric weights can be canceled, which yields a superiorly stable method for computing the simplified barycentric weights, and leads to a fast implementation of the higher order Hermite-Fejér interpolation with linear operations on the number of grids. In addition, the convergence rates are derived for Hermite-Fejér interpolation at Gauss-Jacobi pointsystems.

Key words. Hermite-Fejér interpolation, barycentric, Jacobi polynomial, Gauss-Jacobi point, Lobatto-Gauss-Jacobi point, Chebyshev point.

AMS subject classifications. 65D05, 65D25

1. Introduction. There are many investigations for the behavior of continuous functions approximated by polynomials. Weierstrass [58] in 1885 proved the well known result that every continuous function $f(x)$ in $[-1, 1]$ can be uniformly approximated as closely as desired by a polynomial function. This result has both practical and theoretical relevance, especially in polynomial interpolation.

Polynomial interpolation is a fundamental tool in many areas of numerical analysis. Lagrange interpolation is a well known, classical technique for approximation of continuous functions. Let us denote by

$$(1.1) \quad x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$$

the n distinct points in the interval $[-1, 1]$ and let $f(x)$ be a function defined in the same interval. The n th Lagrange interpolation polynomial of $f(x)$ is uniquely defined by the formula

$$(1.2) \quad L_n[f] = \sum_{k=1}^n f(x_k^{(n)}) \ell_k^{(n)}(x), \quad \ell_k^{(n)}(x) = \frac{\omega_n(x)}{\omega'_n(x_k^{(n)})(x - x_k^{(n)})},$$

where $\omega_n(x) = (x - x_1^{(n)})(x - x_2^{(n)}) \cdots (x - x_n^{(n)})$. However, for an arbitrarily given system of points $\{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\}_{n=1}^\infty$, Bernstein [2] and Faber [13], in 1914, respectively, showed that there exists a continuous function $f(x)$ in $[-1, 1]$ for which the sequence $L_n[f]$ ($n = 1, 2, \dots$) is not uniformly convergent to f in $[-1, 1]$ ¹. Additionally, Bernstein [3] proved that there exists a continuous function $f(x)$ also for which the sequence $L_n[f]$ is divergent. Particularly, Grünwald [23] in 1935 and Marcinkiewicz [32] in 1937, independently, showed that even for the Chebyshev points of first kind

$$(1.3) \quad x_k^{(n)} = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \dots, n, \quad n = 1, 2, \dots,$$

there is a continuous function $f(x)$ in $[-1, 1]$ for which the sequence $L_n[f]$ is divergent everywhere in $[-1, 1]$.

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[†]Department of Applied Mathematics and Software, Central South University, Changsha, Hunan 410083, P. R. China. [‡]Corresponding author

¹A very simple proof was given by Fejér [15] in 1930.

1.1. (Higher order) Hermite-Fejér interpolation. One of the proofs of Weierstrass approximation theorem using interpolation polynomials was presented by Fejér [14] in 1916 based on the above Chebyshev pointsystem (1.3): If $f \in C[-1, 1]$, then there is a unique polynomial $H_{2n-1}(f, x)$ of degree at most $2n-1$ such that $\lim_{n \rightarrow \infty} \|H_{2n-1}(f) - f\|_\infty = 0$, where $H_{2n-1}(f, x)$ is determined by

$$(1.4) \quad H_{2n-1}(f, x_k^{(n)}) = f(x_k^{(n)}), \quad H'_{2n-1}(f, x_k^{(n)}) = 0, \quad k = 1, 2, \dots, n.$$

This polynomial is known as the Hermite-Fejér interpolation polynomial.

The convergence result has been extended to general Hermite-Fejér interpolation of $f(x)$ at nodes (1.1), upon strongly normal pointsystems introduced by Fejér [16]: Given, respectively, the function values $f(x_1^{(n)}), f(x_2^{(n)}), \dots, f(x_n^{(n)})$ and derivatives $d_1^{(n)}, d_2^{(n)}, \dots, d_n^{(n)}$ at these grids, the Hermite-Fejér interpolation polynomial $H_{2n-1}(f)$ has the form of

$$(1.5) \quad H_{2n-1}(f, x) = \sum_{k=1}^n f(x_k^{(n)}) h_k^{(n)}(x) + \sum_{k=1}^n d_k^{(n)} b_k^{(n)}(x),$$

where $h_k^{(n)}(x) = v_k^{(n)}(x) \left(\ell_k^{(n)}(x) \right)^2$, $b_k^{(n)}(x) = (x - x_k^{(n)}) \left(\ell_k^{(n)}(x) \right)^2$ and

$$v_k^{(n)}(x) = 1 - (x - x_k^{(n)}) \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})} \quad (\text{see Fejér [17]}).$$

The pointsystem (1.1) is called strongly normal if for all n

$$(1.6) \quad v_k^{(n)}(x) \geq c > 0, \quad k = 1, 2, \dots, n, \quad x \in [-1, 1]$$

for some positive constant c . The pointsystem (1.1) is called normal if for all n

$$(1.7) \quad v_k^{(n)}(x) \geq 0, \quad k = 1, 2, \dots, n, \quad x \in [-1, 1].$$

Fejér [16] (also see Szegő [45, pp 339]) showed that for the zeros of Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ of degree n ($\alpha > -1$, $\beta > -1$)

$$v_k^{(n)}(x) \geq \min\{-\alpha, -\beta\} \quad \text{for } -1 < \alpha \leq 0, -1 < \beta \leq 0, k = 1, 2, \dots, n \text{ and } x \in [-1, 1].$$

While for the Legendre-Gauss-Lobatto pointsystem (the roots of $(1 - x^2)P_{n-2}^{(1,1)}(x) = 0$),

$$v_k^{(n)}(x) \geq 1, \quad k = 1, 2, \dots, n, \quad x \in [-1, 1].$$

This result is extended to Jacobi-Gauss-Lobatto pointsystem (the roots of $(1 - x^2)P_{n-2}^{(\alpha, \beta)} = 0$) and Jacobi-Gauss-Radau pointsystem (the roots of $(1 - x)P_{n-1}^{(\alpha, \beta)} = 0$ or $(1 + x)P_{n-1}^{(\alpha, \beta)} = 0$) by Vértesi [52, 53]: for all k and $x \in [-1, 1]$,

$$v_k^{(n)}(x) \geq \min\{2 - \alpha, 2 - \beta\} \quad \text{for } \{x_k^{(n-2)}\} \cup \{-1, 1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } 1 \leq \beta \leq 2,$$

$$v_k^{(n)}(x) \geq \min\{2 - \alpha, -\beta\} \quad \text{for } \{x_k^{(n-1)}\} \cup \{1\} \text{ with } 1 \leq \alpha \leq 2 \text{ and } -1 < \beta \leq 0,$$

$$v_k^{(n)}(x) \geq \min\{-\alpha, 2 - \beta\} \quad \text{for } \{x_k^{(n-1)}\} \cup \{-1\} \text{ with } -1 < \alpha \leq 0 \text{ and } 1 \leq \beta \leq 2.$$

Based upon the (strongly) normal pointsystem, Grünwald [24] in 1942 showed that for every $f \in C[-1, 1]$, $\lim_{n \rightarrow \infty} \|H_{2n-1}(f) - f\|_\infty = 0$ if $\{x_k^{(n)}\}$ is strongly normal satisfying (1.6) and $\{d_k^{(n)}\}$ satisfies

$$|d_k^{(n)}| < n^{c-\delta} \quad \text{for some given positive number } \delta, \quad k = 1, 2, \dots, \quad n = 1, 2, \dots,$$

while $\lim_{n \rightarrow \infty} \|H_{2n-1}(f) - f\|_\infty = 0$ in $[-1 + \epsilon, 1 - \epsilon]$ for each fixed $0 < \epsilon < 1$ if $\{x_k^{(n)}\}$ is normal and $\{d_k^{(n)}\}$ is uniformly bounded for $n = 1, 2, \dots$

To get fast convergence on suitable smooth functions, higher order Hermite-Fejér interpolation polynomials were considered in Goodenough and Mills [21], Sharma and Tzimbalaro [39], Szabados [44], Vértési [55], etc.: for $p = 0, 1, \dots, m-1$ and $q = 1, 2, \dots, n$,

$$(1.8) \quad H_{mn-1}(f, x) = \sum_{k=1}^n \sum_{j=0}^{m-1} f^{(j)}(x_k^{(n)}) A_{jk}(x), \quad H_{mn-1}^{(p)}(f, x_q^{(n)}) = f^{(p)}(x_q^{(n)}),$$

where the polynomial $A_{jk}(x)$ of degree at most $mn-1$ satisfies

$$(1.9) \quad A_{jk}^{(p)}(x_q^{(n)}) = \delta_{jp} \delta_{kq}, \quad j = 0, 1, \dots, m-1, \quad k = 1, 2, \dots, n,$$

and δ is the Kronecker delta function. For simplicity, in the following we abbreviate $x_k^{(n)}$ as x_k , $\ell_k^{(n)}(x)$ as $\ell_k(x)$, $h_k^{(n)}(x)$ as $h_k(x)$, and $b_k^{(n)}(x)$ as $b_k(x)$.

The convergences of the higher order Hermite-Fejér interpolation polynomials have been extensively studied (see e.g. Byrne et al. [9], Goodenough and Mills [22], Locher [30], Mathur and Saxena [31], Moldovan [33], Nevai and Vértési [34], Popoviciu [35], [39], Shi [40, 41], Shisha et al. [42], Sun [43], Szili [46], Vecchia et al. [12], Vértési [52, 54] etc.). The convergence rates are achieved most on Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems. As is well known in approximation theory, the right approach is to use point sets that are clustered at the endpoints of the interval with an asymptotic density proportional to $(1-x^2)^{-1/2}$ as $n \rightarrow \infty$ emphasised, for example, in Berrut and Trefethen [5] and Trefethen [48]. Hence, in this paper we confine ourselves to Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems.

1.2. Barycentric forms and implementation on general Hermite interpolation.

In general, the Hermite interpolation is to find a polynomial $H_{N-1}(f, x)$ of degree at most $N-1$ such that

$$\left. \frac{d^r}{dx^r} H_{N-1}(f, x) \right|_{x=x_k} = f_{k,r} \quad \text{for } r = 0, 1, \dots, n_k - 1,$$

where $f_{k,0}, f_{k,1}, \dots, f_{k,n_k-1}$ denote the function value and its first n_k-1 derivatives at the interpolation grid points x_k ($k = 1, 2, \dots, n$), respectively, and $N = n_1 + n_2 + \dots + n_n$.

The polynomial $H_{N-1}(f, x)$ can be represented in either the Newton form or the barycentric form. In the Newton form, the grid points x_k must be ordered in a special way (see Schneider and Warner [38]). If the grid points are not carefully ordered, the Newton form is susceptible to catastrophic numerical instability. For more details, see Fischer and Reichel [18], Tal-Ezer [47], Berrut and Trefethen [5], Butcher et al. [8] and Sadiq and Viswanath [36]. In contrast, the barycentric form does not depend on the order in which the nodes are arranged, which treats all the grid points equally. Barycentric interpolation is arguably the method of choice for numerical polynomial interpolation.

The first barycentric formula for the Hermite interpolation is of the form of

$$(1.10) \quad \begin{aligned} H_{N-1}(f, x) = & H^*(f, x) \sum_{k=1}^n \frac{f_{k,n_k-1}}{(n_k-1)!} \left(\frac{w_{k,0}}{x-x_k} \right) + \frac{f_{k,n_k-2}}{(n_k-2)!} \left(\frac{w_{k,0}}{(x-x_k)^2} + \frac{w_{k,1}}{x-x_k} \right) \\ & + \dots + f_{k,0} \left(\frac{w_{k,0}}{(x-x_k)^{n_k}} + \dots + \frac{w_{k,n_k-1}}{x-x_k} \right), \end{aligned}$$

where $H_N^*(f, x) = \prod_{k=1}^n (x-x_k)^{n_k}$ and $w_{k,r}$ is called the barycentric weights. Applying $1 \equiv H_N^*(f, x) \sum_{k=1}^n \sum_{r=0}^{n_k-1} w_{k,r} (x-x_k)^{r-n_k}$ derives the second barycentric form²

$$(1.11) \quad H_{N-1}(f, x) = \frac{\sum_{k=1}^n \sum_{s=0}^{n_k-1} \frac{f_{k,s}}{s!} \sum_{r=0}^{n_k-s-1} w_{k,r} (x-x_k)^{r+s-n_k}}{\sum_{k=1}^n \sum_{r=0}^{n_k-1} w_{k,r} (x-x_k)^{r-n_k}}$$

²Two typos occur in (1.4) [36]: $(x-x_k)^{n_k-r-s}$ should be $(x-x_k)^{r+s-n_k}$ and $(x-x_k)^{n_k-r}$ be $(x-x_k)^{r-n_k}$.

(see e.g. [8, 36]).

The second barycentric form is more robust in the presence of rounding errors in the weights $w_{k,r}$. It is obvious from inspection that either of the two forms can be used to evaluate the interpolant $H_{N-1}(f, x)$ at a given point x using $O(\sum_{k=1}^n n_k^2)$ arithmetic operations once the barycentric weights $w_{k,r}$ are known.

Schneider and Werner [38] used divided differences to evaluate the barycentric weights. This method requires $n(n-1)/2$ divisions, $(N^2 - \sum_{k=1}^n n_k^2)/2$ multiplications and about the same number of subtractions or additions [36]. However, the numerical stability depends upon a good ordering of the grid points as mentioned above. Moreover, Newton interpolation requires the recomputation of the divided difference tableau for each new function.

Butcher et al. [8] introduced an efficient method, compared with that in [38], for computing the barycentric weights, which is derived by using contour integrals and the manipulation of infinite series. More recently, Sadiq and Viswanath [36] gave another more direct and simple derivation of this method: Calculating the barycentric weights is to find the coefficients in the Taylor polynomial of expressions of the form

$$\prod_{j \neq k} (x + x_k - x_j)^{-n_j} = \prod_{j \neq k} (x_k - x_j)^{-n_j} \prod_{j \neq k} \left(1 - \frac{x - x_k}{x_k - x_j}\right)^{-n_j}$$

which can be obtained by the following recursion

$$I_{k,r} = \sum_{s=1}^r P_{k,s} I_{k,r-s}/r, \quad I_{k,0} = 1, \quad P_{k,r} = \sum_{j \neq k} n_j (x_j - x_k)^{-r}, \quad r = 1, 2, \dots, n_k - 1,$$

and then $w_{k,r} = C_k I_{k,r}$ with $C_k = \prod_{j \neq k} (x_k - x_j)^{-n_j}$. It costs $O(n \sum_{k=1}^n n_k + \sum_{k=1}^n n_k^2)$ multiplications. Roughly half these operations are additions or subtractions and roughly half are multiplications. Furthermore, if an additional derivative is prescribed at one of the interpolation points, update the barycentric coefficients use only $O(N)$ operations [36].

Notice that the barycentric Hermite interpolation problem is highly susceptible to overflows or underflows. The weights $w_{k,r}$ in (1.10) and (1.11) usually vary by exponentially large factors. Figures 1.1-1.2 illustrate the magnitudes of the barycentric weights computed by the method of Sadiq and Viswanath [36] at the Chebyshev pointsystem (1.3) or Legendre pointsystem with different multiple number m of derivatives ($n_1 = n_2 = \dots = n_n = m$). We can see from these two figures that the barycentric weights become extremely large while the number of points and the multiple number of derivatives are not so large, which will lead to overflows³ for larger n or m . Table 1.1 shows the threshold S that the algorithm suffers overflows for computation of the weights if $n \geq S$ with different m , respectively.

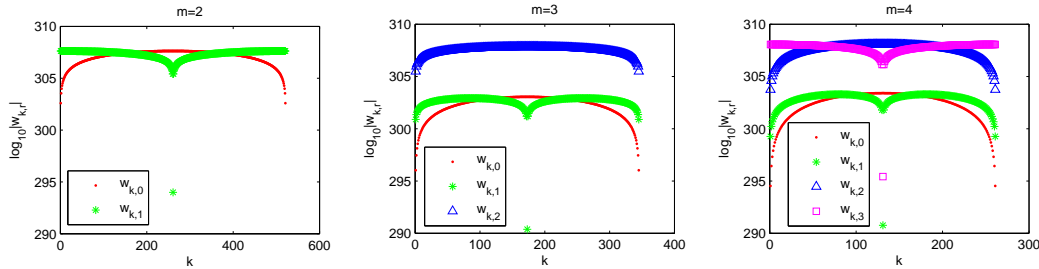


FIG. 1.1. Magnitudes of the barycentric weights $w_{k,r}$ by method of Sadiq and Viswanath interpolating at Chebyshev pointsystem (1.3).

³In MATLAB, the largest positive normalized floating-point number in IEEE double precision is $(1 + (1 - 2^{-52}))2^{1023} \approx 1.7977 \times 10^{308}$, the smallest positive normalized floating-point number in IEEE double precision is $2^{-1022} \approx 2.225 \times 10^{-308}$.

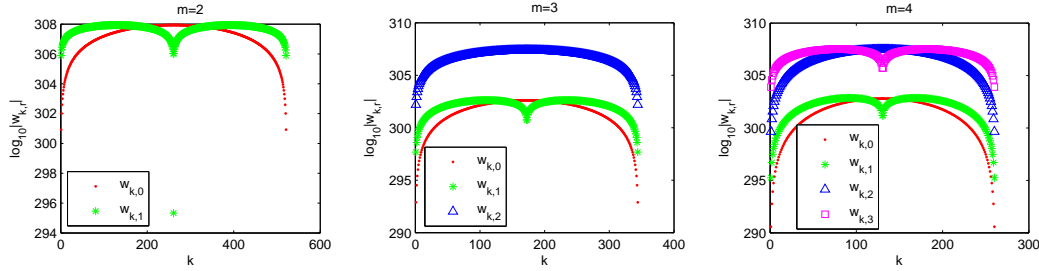


FIG. 1.2. Magnitudes of the barycentric weights $w_{k,r}$ by method of Sadiq and Viswanath interpolating at Legendre pointsystem.

TABLE 1.1

The threshold S that the algorithm [36] collapses for computation of the barycentric weights if $n \geq S$ with different m at the Chebyshev pointsystem (1.3) and Legendre pointsystem

| Chebyshev pointsystem (1.3) | | | Legendre pointsystem | | |
|-----------------------------|---------|---------|----------------------|---------|---------|
| $m = 2$ | $m = 3$ | $m = 4$ | $m = 2$ | $m = 3$ | $m = 4$ |
| $S=524$ | $S=347$ | $S=263$ | $S=523$ | $S=346$ | $S=262$ |

Thus, in [36], Sadiq and Viswanath used $2 \cos((2k-1)\pi/(2n))$ instead of $\cos((2k-1)\pi/(2n))$ and considered Leja reordering of the points to get more stable computation of the weights and decrease the chance of overflows or underflows. However, reordering of the points does not change the magnitudes of the barycentric weights. Furthermore, Leja reordering needs $O(n^2)$ operations [10].

Fortunately, from the second barycentric form (1.11), we see that the weights $w_{k,r}$ appear in the denominator exactly as in the numerator. Due to the division, the barycentric weights can be simplified by cancelling the common factors without altering the result (see Figures 2.1-2.4 and Tables 2.1-2.4 below).

In this paper, we are concerned with fast implementation of the higher order Hermite-Fejér interpolation polynomial (1.8), based on the second barycentric form (1.11), at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems.

Recently, a new algorithm on the evaluation of the nodes and weights for the Gauss quadrature was given by Glaser, Liu and Rokhlin [20] with $O(n)$ operations, which has been extended by both Bogaert, Michiels and Fostier [7], and Hale and Townsend [25]. A MATLAB routine for computation of these nodes and weights can be found in CHEBFUN system [51].

As a result of these developments, in Section 2, we will discuss in details on calculation of the second barycentric weights of the higher order Hermite-Fejér interpolation polynomial (1.11) at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems, and present two algorithms with $O(nm^2)$ operations. Particularly, due to division, the common factor in the barycentric weights can be canceled, which yields a superiorly stable method for computing the simplified barycentric weights. In Section 3, we will consider the stabilities of these implementations for the second barycentric formula on higher order Hermite-Fejér interpolation and present numerical examples illustrating the efficiency and accuracy. A final remark on the convergence rate of Hermite-Fejér interpolation (1.5) is included in Section 4.

All the numerical results in this paper are carried out by using MATLAB R2012a on a desktop (2.8 GB RAM, 2 Core2 (32 bit) processors at 2.80 GHz) with Windows XP operating system.

2. Fast computation of the barycentric weights on higher order Hermite-Fejér interpolation. In this section, we will introduce two methods for fast computation of the barycentric weights on higher order Hermite-Fejér interpolation at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems, which both share $O(nm^2)$ operations and lead to fast and stable calculation of the barycentric weights due to the division in (1.11), where the exponentially increasing common factor is cancelled.

Algorithm 1: Following the barycentric Hermite interpolation formula in [8, 36, 38], we rewrite the higher order Hermite-Fejér interpolation (1.8) as

$$(2.1) \quad H_{mn-1}(f, x) = H_{mn}^*(x) \sum_{k=1}^n \frac{f_{k,m-1}}{(m-1)!} \left(\frac{w_{k,0}}{x-x_k} \right) + \frac{f_{k,m-2}}{(m-2)!} \left(\frac{w_{k,0}}{(x-x_k)^2} + \frac{w_{k,1}}{x-x_k} \right) \\ + \cdots + f_{k,0} \left(\frac{w_{k,0}}{(x-x_k)^m} + \cdots + \frac{w_{k,m-1}}{x-x_k} \right)$$

with

$$(2.2) \quad \left. \frac{d^r H_{mn-1}(f, x)}{dx^r} \right|_{x=x_k} = f_{k,r} \quad \text{for } r = 0, \dots, m-1 \text{ and } k = 1, \dots, n,$$

where $H_{mn}^*(x) = \prod_{k=1}^n (x-x_k)^m$ and $f_{k,j} = f^{(j)}(x_k)$. Denote by $\hbar_k(x) = \frac{\omega_n(x)}{x-x_k} = \omega_n'(x_k) \ell_k(x)$, then, expression (2.1) can be represented as

$$(2.3) \quad H_{mn-1}(f, x) = \sum_{k=1}^n f_{k,0} \left(w_{k,0} \hbar_k^m(x) + w_{k,1} \hbar_k^m(x)(x-x_k) + \cdots + w_{k,m-1} \hbar_k^m(x)(x-x_k)^{m-1} \right) \\ + \cdots + \frac{f_{k,m-1}}{(m-1)!} w_{k,0} \hbar_k^m(x)(x-x_k)^{m-1}.$$

Furthermore, from (1.9) and (2.1)-(2.3), we have

$$(2.4) \quad \left. w_{k,0} \hbar_k^m(x) \right|_{x=x_k} = 1, \\ \left. w_{k,0} (\hbar_k^m(x))^{(j)} + w_{k,1} (\hbar_k^m(x)(x-x_k))^{(j)} + \cdots + w_{k,j} (\hbar_k^m(x)(x-x_k)^j)^{(j)} \right|_{x=x_k} = 0$$

for $j = 1, \dots, m-1$. By using the Taylor expansion of $\hbar_k^m(x)$ at x_k , it leads to

$$\left. \frac{(\hbar_k^m(x)(x-x_k)^i)^{(j)}}{j!} \right|_{x=x_k} = \left. \frac{(\hbar_k^m(x))^{(j-i)}}{(j-i)!} \right|_{x=x_k}, \quad i = 0, 1, \dots, j-1.$$

Thus, from (2.4), we get the following formulas

$$(2.5) \quad w_{k,0} = \frac{1}{(\omega_n'(x_k))^m}, \quad w_{k,j} = - \sum_{i=0}^{j-1} \frac{w_{k,i}}{(j-i)!} \left(\ell_k^m(x) \right)^{(j-i)} \Big|_{x=x_k}, \quad j = 1, 2, \dots, m-1.$$

In addition, from the definition of $\ell_k(x) = \frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}$, it is not difficult to deduce by applying the Taylor expansion of $\omega_n(x)$ at $x = x_k$ that

$$(2.6) \quad \frac{(\ell_k(x))^{(r)}}{r!} \Big|_{x=x_k} = \frac{\omega_n^{(r+1)}(x_k)}{(r+1)! \omega_n'(x_k)} := M_{k,r}, \quad r = 0, \dots, m-1.$$

Set $b_{k,j} = \frac{[\ell_k^m(x)]_{x=x_k}^{(j)}}{j!}$ ($j = 0, 1, \dots, m-1$) and $a_{k,i} = \frac{m}{(i-1)!} [\ell_k'(x)]_{x=x_k}^{(i-1)}$ ($i = 1, 2, \dots, m-1$).

Noting that $b_{k,j} = \frac{m}{j!} [\ell_k^{m-1}(x) \ell_k'(x)]_{x=x_k}^{(j-1)} = \frac{m}{j!} [\ell_k^m(x) \frac{\ell_k'(x)}{\ell_k(x)}]_{x=x_k}^{(j-1)}$, it follows by Leibniz formula that

$$(2.7) \quad b_{k,j} = \frac{1}{j} \sum_{i=1}^j a_{k,i} b_{k,j-i}, \quad j = 1, 2, \dots, m-1; \quad b_{k,0} = 1.$$

In the following, we shall show that $a_{k,i}$ can be computed from $M_{k,r}$, then $b_{k,j}$ can be evaluated from the recursion (2.7). In fact, $a_{k,i}$ can be calculated from the coefficient of the Taylor expansion of $\ell_k'(x)/\ell_k(x)$ at $x = x_k$ by the following lemma.

LEMMA 2.1. Let $F(x) = \frac{A(x)}{B(x)} = \sum_{j=1}^s F_j(x-x_k)^{j-1} + O((x-x_k)^s)$, where $A(x) = \sum_{j=1}^s A_j(x-x_k)^{j-1} + O((x-x_k)^s)$ and $B(x) = \sum_{j=1}^s B_j(x-x_k)^{j-1} + O((x-x_k)^s)$, then it

obtains

$$(2.8) \quad B_1 F_j = A_j - \sum_{i=2}^j B_i F_{j-i+1}, \quad j = 1, \dots, s.$$

Proof. The proof is trivial. \square
Since

$$(2.9) \quad \ell_k(x) = \sum_{r=1}^{m-1} M_{k,r-1}(x-x_k)^{r-1} + O((x-x_k)^{m-1}),$$

and

$$(2.10) \quad \ell'_k(x) = \sum_{r=1}^{m-1} r M_{k,r}(x-x_k)^{r-1} + O((x-x_k)^{m-1}),$$

by Lemma 2.1 and using $M_{k,0} = 1$ and $\frac{\ell'_k(x)}{\ell_k(x)} = \frac{1}{m} \sum_{r=1}^{m-1} a_{k,r}(x-x_k)^{r-1} + O((x-x_k)^{m-1})$, we have

$$(2.11) \quad a_{k,1} = m M_{k,1}, \quad a_{k,i} = i m M_{k,i} - \sum_{j=2}^i M_{k,j-1} a_{k,i-j+1}, \quad i = 2, \dots, m-1,$$

and then by (2.5) we get

$$(2.12) \quad w_{k,0} = \frac{1}{(\omega'(x_k))^m}, \quad w_{k,j} = - \sum_{i=0}^{j-1} w_{k,i} b_{k,j-i}, \quad j = 1, \dots, m-1.$$

Thus, if $\{M_{k,r}\}_{r=0}^{m-1}$ and $w_{k,0}$ are known, from (2.11) the total computation of $\{a_{k,j}\}_{j=0}^{m-1}$ costs $O(m^2)$ operations, the same as those for $\{b_{k,j}\}_{j=0}^{m-1}$ and $\{w_{k,j}\}_{j=0}^{m-1}$ by (2.7) and (2.12), respectively, and then the implementation of (1.11) with $n_1 = \dots = n_n = m$ costs $O(nm^2)$ operations.

Algorithm 2: The barycentric weights $w_{k,r}$ of the Hermite-Fejér interpolation (2.1) can also be calculated by another fast way based on the formula given in Szabados [44],

$$(2.13) \quad H_{mn-1}(f, x) = \sum_{k=1}^n \sum_{j=0}^{m-1} f^{(j)}(x_k) \frac{\ell_k(x)^m}{j!} \sum_{i=0}^{m-j-1} \frac{[\ell_k(x)^{-m}]_{x=x_k}^{(i)}}{i!} (x-x_k)^{i+j}.$$

We rewrite the interpolation (2.13) in the first barycentric interpolation form

$$(2.14) \quad H_{mn-1}(f, x) = H_{mn}^*(x) \sum_{k=1}^n \frac{1}{(\omega'_n(x_k))^m} \sum_{j=0}^{m-1} \frac{f_{k,j}}{j!} \sum_{i=0}^{m-j-1} \frac{[\ell_k(x)^{-m}]_{x=x_k}^{(i)}}{i!} (x-x_k)^{i+j-m},$$

and second barycentric interpolation form

$$(2.15) \quad H_{mn-1}(f, x) = \frac{\sum_{k=1}^n w_{k,0} \sum_{j=0}^{m-1} \frac{f_{k,j}}{j!} \sum_{i=0}^{m-j-1} \frac{[\ell_k(x)^{-m}]_{x=x_k}^{(i)}}{i!} (x-x_k)^{i+j-m}}{\sum_{k=1}^n w_{k,0} \sum_{i=0}^{m-1} \frac{[\ell_k(x)^{-m}]_{x=x_k}^{(i)}}{i!} (x-x_k)^{i-m}}$$

respectively. Next we concentrate on the computation of $\tilde{b}_{k,j} := \frac{[\ell_k(x)^{-m}]_{x=x_k}^{(j)}}{j!}$ ($j = 0, 1, \dots, m-1$). Comparing (2.14) with (2.1), we find that the barycentric weights satisfies

$$(2.16) \quad w_{k,j} = \tilde{b}_{k,j} w_{k,0}, \quad j = 0, 1, \dots, m-1.$$

From [44], it follows

$$(2.17) \quad \tilde{b}_{k,j} = \frac{1}{j} \sum_{i=1}^j \tilde{a}_{k,i} \tilde{b}_{k,j-i}, \quad j = 1, 2, \dots, m-1; \quad \tilde{b}_{k,0} = 1,$$

where $\tilde{a}_{k,i} = \frac{m}{(i-1)!} \left[\frac{1}{x-x_k} - \frac{\omega'_n(x)}{\omega_n(x)} \right]_{x=x_k}^{(i-1)}$ ($i = 1, \dots, m-1$). Then from $\omega_n(x) = \omega'_n(x_k) \ell_k(x)(x - x_k)$, it is easy to see that $\tilde{a}_{k,i} = \frac{m}{(i-1)!} \left[\frac{\ell_k(x) \omega'_n(x_k) - \omega'_n(x)}{\omega_n(x)} \right]_{x=x_k}^{(i-1)} = \frac{m}{(i-1)!} \left[\frac{-\ell'_k(x)}{\ell_k(x)} \right]_{x=x_k}^{(i-1)}$. Similarly, by Lemma 2.1 and (2.9)-(2.10), we get

$$(2.18) \quad \tilde{a}_{k,j} = -mM_{k,1}, \quad \tilde{a}_{k,j} = -jmM_{k,j} - \sum_{i=2}^j M_{k,j-1} \tilde{a}_{k,j-i+1}, \quad j = 2, \dots, m-1.$$

Thus, from (2.18) and (2.17), the total computation of barycentric weights $\{\tilde{b}_{k,r}\}_{r=0}^{m-1}$ costs also $O(m^2)$ operations if $\{M_{k,r}\}_{r=0}^{m-1}$ and $w_{k,0}$ are known.

From the above illustrations, we see that fast computation of $M_{k,r}$ and $w_{k,0}$ leads to fast implementation of higher order barycentric Hermite-Fejer interpolation. We shall show that for each k , $\{M_{k,r}\}_{r=0}^{m-1}$ can be rapidly calculated with $O(m)$ operations for Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems.

2.1. Gauss-Jacobi pointsystems. Let $\{x_k\}_{k=1}^n$ be the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Thus $\omega_n(x) = \frac{P_n^{(\alpha,\beta)}(x)}{K_n}$, where K_n is the leading coefficient of $P_n^{(\alpha,\beta)}(x)$. From (2.6), we get

$$M_{k,r-1} = \frac{\omega_n^{(r)}(x_k)}{r! \omega'_n(x_k)} = \frac{\frac{d^r P_n^{(\alpha,\beta)}}{dx^r}(x_k)}{r! P_n^{(\alpha,\beta)}(x_k)}, \quad r = 1, \dots, m.$$

It is known that $P_n^{(\alpha,\beta)}(x)$ is the unique solution of the second order linear homogeneous Sturm-Liouville differential equation

$$(2.19) \quad (1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0,$$

from which it is not difficult to deduce that

$$(2.20) \quad (1-x^2)y^{(r+2)} + [\beta - \alpha - (\alpha + \beta + 2(r+1))x]y^{(r+1)} + [n(n + \alpha + \beta + 1) - r(\alpha + \beta + r + 1)]y^{(r)} = 0,$$

for $r = 0, 1, \dots$. Thus, we have

$$(2.21) \quad M_{k,r+1} = \frac{(\alpha + \beta + 2(r+1))x_k + \alpha - \beta}{1-x_k^2} \frac{1}{(r+2)} M_{k,r} + \frac{r(\alpha + \beta + r + 1) - n(n + \alpha + \beta + 1)}{1-x_k^2} \frac{1}{(r+2)(r+1)} M_{k,r-1},$$

with $M_{k,0} = 1$, $M_{k,1} = \frac{\alpha - \beta + (\alpha + \beta + 2)x_k}{2(1-x_k^2)}$ and then $\{M_{k,r}\}_{r=0}^{m-1}$ can be computed in $O(m)$. Consequently, from (2.6)-(2.7), (2.11)-(2.12), (2.14)-(2.15) and (2.18), the barycentric form (1.11) with $n_1 = \dots = n_n = m$ and (2.15) can be achieved in $O(nm^2)$ operations if $w_{k,0}$ is known.

From Wang et al. [57], $w_{k,0}$ has the explicit form

$$(2.22) \quad w_{k,0} = \left[C_n^{(\alpha,\beta)}(-1)^{k+1} \sqrt{(1-x_k^2)\overline{w}_k} \right]^m, \quad k = 1, 2, \dots, n,$$

where \overline{w}_k is the Gaussian quadrature weight corresponding to x_k for the Jacobi weight,

$$C_n^{(\alpha,\beta)} = \sigma_n \frac{\Gamma(2n + \alpha + \beta + 1)}{2^{n+\frac{\alpha+\beta+1}{2}}} \frac{1}{\sqrt{n! \Gamma(n + \alpha + \beta + 1) \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}}$$

and $\sigma_n = +1$ for n odd and $\sigma = -1$ for n even. Moreover, both $\{x_k\}_{k=1}^n$ and $\{\overline{w}_k\}_{k=1}^n$ can be efficiently calculated by routine **jacpts** in CHEBFUN [51] with $O(n)$ operations.

Additionally, it is worth noting that due to the division of the second barycentric form (1.11) or formula (2.15), the common factor $(C_n^{(\alpha,\beta)})^m$ of weights $w_{k,r}$ from (2.12) and (2.16) can be cancelled without affecting the value of $H_{mn-1}(f, x)$ (we call these new weights as simplified barycentric weights). Then the barycentric weight $w_{k,0}$ can be simplified as

$$(2.23) \quad w_{k,0} = (-1)^{m(k+1)} \left[\sqrt{(1-x_k^2)\overline{w}_k} \right]^m.$$

Comparing these two algorithms with Sadiq and Viswanath's [36], we find that the new algorithms cost $O(nm^2)$ operations much less than that $O(n^2m + nm^2)$ given by Sadiq and Viswanath [36] if $n \gg m$. Moreover, due to the cancellation of the common factor $(C_n^{(\alpha,\beta)})^m$, the computation of $\{w_{k,j}\}_{j=0}^{m-1}$ is quite efficient and stable (see Figures 2.1-2.2 and Tables 2.3-2.4). Tables 2.1-2.2 show the values of the common factor $(C_n^{(\alpha,\beta)})^m$ with respect to Chebyshev pointsystem (1.3) and Gauss-Legendre pointsystem, respectively.

TABLE 2.1
 $(C_n^{(\alpha,\beta)})^m$ with respect to Chebyshev pointsystem (1.3): $\alpha = \beta = -0.5$

| | $m = 2$ | $m = 3$ | $m = 4$ | $m = 10$ |
|------------|----------------------|-----------------------|-----------------------|-----------------------|
| $n = 100$ | $1.27876 * 10^{57}$ | $-4.57282 * 10^{85}$ | $1.63523 * 10^{114}$ | $3.41937 * 10^{285}$ |
| $n = 200$ | $1.02744 * 10^{117}$ | $-3.29335 * 10^{175}$ | $1.05564 * 10^{234}$ | $1.14496 * 10^{585}$ |
| $n = 500$ | $1.70536 * 10^{297}$ | $-7.04245 * 10^{445}$ | $2.90825 * 10^{594}$ | $1.44238 * 10^{1486}$ |
| $n = 1000$ | $9.13653 * 10^{597}$ | $-8.73318 * 10^{896}$ | $8.34762 * 10^{1195}$ | $6.36660 * 10^{2989}$ |

TABLE 2.2
 $(C_n^{(\alpha,\beta)})^m$ with respect to Gauss-Legendre pointsystem: $\alpha = \beta = 0$

| | $m = 2$ | $m = 3$ | $m = 4$ | $m = 10$ |
|------------|----------------------|-----------------------|-----------------------|-----------------------|
| $n = 100$ | $2.55114 * 10^{57}$ | $-1.28855 * 10^{86}$ | $6.50829 * 10^{114}$ | $1.08061 * 10^{287}$ |
| $n = 200$ | $2.05232 * 10^{117}$ | $-9.29755 * 10^{175}$ | $4.21203 * 10^{234}$ | $3.64106 * 10^{586}$ |
| $n = 500$ | $3.40901 * 10^{297}$ | $-1.99041 * 10^{446}$ | $1.16214 * 10^{595}$ | $4.60408 * 10^{1487}$ |
| $n = 1000$ | $1.82685 * 10^{598}$ | $-2.46919 * 10^{897}$ | $3.33738 * 10^{1196}$ | $2.03477 * 10^{2991}$ |

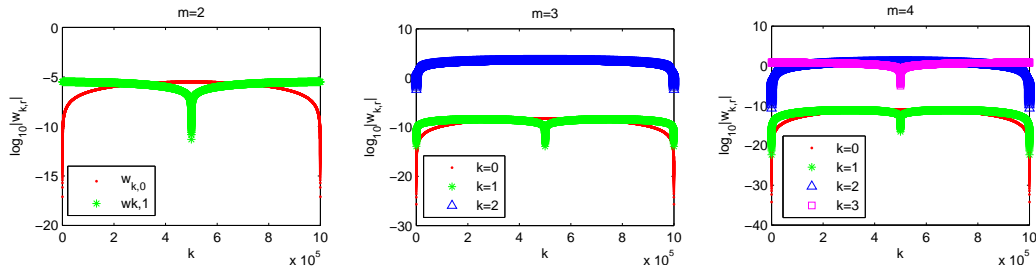


FIG. 2.1. Magnitude of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** interpolating at the Chebyshev pointsystem (1.3): $k = 1 : 10^6$ and $r = 0 : m - 1$.

REMARK 2.2. The simplified barycentric weights (2.23), in the case $m = 1$ ($w_{k,0} = (-1)^{k+1} \sqrt{(1-x_k^2)\overline{w}_k}$), is exact the barycentric weight for the barycentric formula of Lagrange interpolation at the Jacobi pointsystem $\{x_k\}_{k=1}^n$, which was derived for Chebyshev points of second kind in Henrici [27], for Legendre points in Wang and Xiang [56], and extended to Jacobi points in Hale and Trefethen [26].

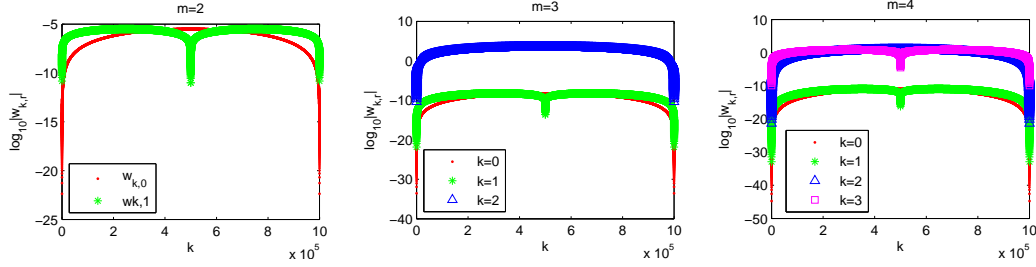


FIG. 2.2. Magnitude of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** interpolating at Legendre pointsystem: $k = 1 : 10^6$ and $r = 0 : m - 1$.

TABLE 2.3

The CPU time for computation of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** at the Chebyshev pointsystem (1.3): $k = 1 : n$ and $r = 0 : m - 1$.

| | $m = 2$ | $m = 3$ | $m = 4$ | $m = 10$ |
|------------|-----------|-----------|-----------|-----------|
| $n = 10^4$ | 0.011127s | 0.012392s | 0.012778s | 0.026214s |
| $n = 10^5$ | 0.031632s | 0.046582s | 0.061328s | 0.271533s |
| $n = 10^6$ | 0.239484s | 0.377926s | 0.530127s | 2.813712s |

TABLE 2.4

The CPU time for computation of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** at the Legendre pointsystem: $k = 1 : n$ and $r = 0 : m - 1$.

| | $m = 2$ | $m = 3$ | $m = 4$ | $m = 10$ |
|------------|------------|------------|------------|------------|
| $n = 10^4$ | 0.166094 | 0.169577s | 0.170217s | 0.182489s |
| $n = 10^5$ | 1.713747s | 1.712082s | 1.734930s | 1.970211s |
| $n = 10^6$ | 17.445534s | 17.745105s | 18.010896s | 20.098777s |

2.2. Jacobi-Gauss-Lobatto pointsystems. Suppose $-1 = x_1 < x_2 < \dots < x_{n-1} < x_n = 1$ are the n zeros of $(x^2 - 1)P_{n-2}^{(\alpha, \beta)}(x)$. Thus from (2.6) we get

$$(2.24) \quad M_{r-1}(x) = \frac{\omega_n^{(r)}(x)}{r! \omega_n'(x)} = \frac{(x^2 - 1) \left(P_{n-2}^{(\alpha, \beta)} \right)^{(r)}(x) + 2rx \left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-1)}(x) + r(r-1) \left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-2)}(x)}{r! \left((x^2 - 1) \left(P_{n-2}^{(\alpha, \beta)} \right)'(x) + 2x P_{n-2}^{(\alpha, \beta)}(x) \right)},$$

for $r = 2, \dots, m$. Especially, $M_{k,0} = M_0(x_k) = 1$ for $k = 1, 2, \dots, n$.

For the case $x_k = \pm 1$, it follows

$$(2.25) \quad M_{1,r-1} = M_{r-1}(-1) = \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-1)}(-1)}{(r-1)! P_{n-2}^{(\alpha, \beta)}(-1)} - \frac{1}{2} \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-2)}(-1)}{(r-2)! P_{n-2}^{(\alpha, \beta)}(-1)}, \quad r = 2, \dots, m,$$

$$(2.26) \quad M_{n,r-1} = M_{r-1}(1) = \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-1)}(1)}{(r-1)! P_{n-2}^{(\alpha, \beta)}(1)} + \frac{1}{2} \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-2)}(1)}{(r-2)! P_{n-2}^{(\alpha, \beta)}(1)}, \quad r = 2, \dots, m,$$

and for the case $x_k \neq \pm 1$ and $r = 2, \dots, m$,

$$(2.27) \quad M_{k,r-1} = M_{r-1}(x_k) = \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r)}(x_k)}{r! \left(P_{n-2}^{(\alpha, \beta)} \right)'(x_k)} + \frac{2x_k}{x_k^2 - 1} \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-1)}(x_k)}{(r-1)! \left(P_{n-2}^{(\alpha, \beta)} \right)'(x_k)} + \frac{1}{x_k^2 - 1} \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-2)}(x_k)}{(r-2)! \left(P_{n-2}^{(\alpha, \beta)} \right)'(x_k)},$$

where $\frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r)}(x_k)}{r! \left(P_{n-2}^{(\alpha, \beta)} \right)'(x_k)}$ can be evaluated by (2.21) with $n - 2$ instead of n for $x_k \neq \pm 1$, and for $x_k = \pm 1$ by

$$\frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r)}(\pm 1)}{r! \left(P_{n-2}^{(\alpha, \beta)} \right)'(\pm 1)} = \frac{(n-2)(\alpha + \beta + n - 1) - (r-1)(\alpha + \beta + r)}{r(\pm(\alpha + \beta + 2r) + \alpha - \beta)} \frac{\left(P_{n-2}^{(\alpha, \beta)} \right)^{(r-1)}(\pm 1)}{(r-1)! \left(P_{n-2}^{(\alpha, \beta)} \right)'(\pm 1)}$$

for $r = 1, 2, \dots, m-1$.

Moreover, from [57], $w_{k,0}$ has the explicit form of

$$(2.28) \quad w_{k,0} = \left[C_{n-2}^{(\alpha,\beta)} (-1)^{k+1} \sqrt{\eta_k \widehat{w}_k} \right]^m, \eta_k = \begin{cases} \beta, & k=1, \\ \alpha, & k=n, \\ 1, & \text{otherwise,} \end{cases}$$

where $\widehat{w}_k = \frac{\overline{w}_k}{1-x_k^2}$ for $k = 2, 3, \dots, n-1$,

$$\widehat{w}_1 = 2^{\alpha+\beta-1} \frac{\Gamma(\beta)\Gamma(\beta+1)\Gamma(n+\alpha+1)n!}{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}, \widehat{w}_n = 2^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(n+\beta+1)n!}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}$$

and $\{\overline{w}_k\}_{k=2}^{n-1}$ is the Gaussian quadrature weight corresponding to $\{x_k\}_{k=2}^{n-1}$. Due to the division in (2.15), the barycentric weight $w_{k,0}$ can be simplified as

$$(2.29) \quad w_{k,0} = (-1)^{m(k+1)} (\eta_k \widehat{w}_k)^{m/2}.$$

Figures 2.3-2.4 show magnitude of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** interpolating at the Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$ and Legendre-Gauss-Lobatto pointsystem with $n = 10^6$ and $m = 2, 3, 4$ respectively.

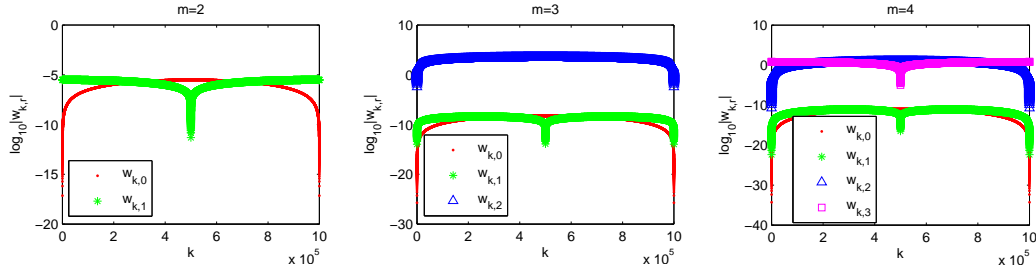


FIG. 2.3. Magnitude of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** interpolating at the Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$: $k = 1 : 10^6$ and $r = 0 : m-1$.

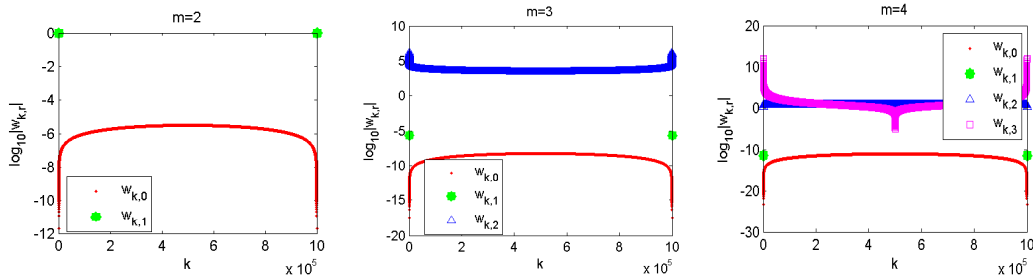


FIG. 2.4. Magnitude of the simplified barycentric weights $w_{k,r}$ by the **Algorithm 2** interpolating at Legendre-Lobatto pointsystem: $k = 1 : 10^6$ and $r = 0 : m-1$. $w_{k,1} = 0$ except the first and the last term.

REMARK 2.3. In the case $m = 1$ and $\alpha = \beta = 0.5$, the nodes $\{x_k\}_{k=1}^n$ are the Chebyshev points of second kinds

$$(2.30) \quad x_k = \cos\left(\frac{k-1}{n-1}\pi\right), \quad k = 1, 2, \dots, n$$

and $w_{k,0} = (-1)^{k+1} \eta_k$ with $\eta_k = \begin{cases} \frac{1}{2}, & k=1, n \\ 1, & \text{otherwise} \end{cases}$ derived by Salzer [37].

Algorithm 1 Barycentric weights at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems

- 1: Input parameters n, m, α, β .
 - 2: Compute the nodes $\{x_k\}_{k=1}^n$ and simplified barycentric weights $\{w_{k,0}\}_{k=1}^n$ by **jacpts**.
 - 3: Compute $M_{k,r} = \frac{\omega_n^{(r+1)}(x_k)}{(r+1)!\omega'_n(x_k)}$ ($k = 1, \dots, n, r = 1, \dots, m-1$) by recursion.
 - 4: Compute $a_{k,i} = imM_{k,i} - \sum_{j=1}^{i-1} a_{k,j}M_{k,i-j}$ ($k = 1, \dots, n, i = 1, \dots, m-1$).
 - 5: Compute $b_{k,i} = \frac{1}{i} \sum_{v=1}^i a_{k,v}b_{k,i-v}$ ($k = 1, \dots, n, i = 1, \dots, m-1$) with $b_{k,0} = 1$.
 - 6: Let $c_{k,0} = 1$, compute $c_{k,i} = -\sum_{j=1}^{i-1} c_{k,j}b_{k,i-j}$ ($k = 1, \dots, n, i = 1, \dots, m-1$).
 - 7: Return $w_{k,i} = w_{k,0}c_{k,i}$ ($k = 1, \dots, n, i = 0, \dots, m-1$).
-

Algorithm 2 Barycentric weights at Gauss-Jacobi or Jacobi-Gauss-Lobatto pointsystems

- 1: Input parameters n, m, α, β .
 - 2: Compute the nodes $\{x_k\}_{k=1}^n$ and simplified barycentric weights $\{w_{k,0}\}_{k=1}^n$ by **jacpts**.
 - 3: Compute $M_{k,r} = \frac{\omega_n^{(r+1)}(x_k)}{(r+1)!\omega'_n(x_k)}$ ($k = 1, \dots, n, r = 1, \dots, m$) by recursion.
 - 4: Compute $\tilde{a}_{k,i} = -imM_{k,i} - \sum_{j=1}^{i-1} \tilde{a}_{k,j}M_{k,i-j}$ ($k = 1, \dots, n, i = 1, \dots, m-1$).
 - 5: Compute $\tilde{b}_{k,i} = \frac{1}{i} \sum_{v=1}^i \tilde{a}_{k,v}\tilde{b}_{k,i-v}$ ($k = 1, \dots, n, i = 1, \dots, m-1$) with $\tilde{b}_{k,0} = 1$.
 - 6: Return $w_{k,i} = w_{k,0}\tilde{b}_{k,i}$ ($k = 1, \dots, n, i = 0, \dots, m-1$).
-

2.3. Lower order Hermite-Fejér interpolation. In particular, from (2.5), the barycentric weights for lower order Hermite-Fejér barycentric interpolation can be given in the explicit forms.

- $m = 2$: $w_{k,1} = -\frac{\omega_n''(x_k)}{\omega_n'(x_k)}w_{k,0}$. Moreover, for the Gauss-Jacobi pointsystem,

$$w_{k,1} = (\beta - \alpha - (\alpha + \beta + 2)x_k)\bar{w}_k, \quad k = 1, 2, \dots, n,$$

while for the Jacobi-Gauss-Lobatto pointsystem,

$$w_{1,1} = \left(1 + \frac{(n-2)(n+\alpha+\beta-1)}{\beta+1}\right)\beta\hat{w}_1, \quad w_{n,1} = \left(-1 - \frac{(n-2)(n+\alpha+\beta-1)}{\alpha+1}\right)\alpha\hat{w}_n$$

and

$$w_{k,1} = (\beta - \alpha - (\alpha + \beta - 2)x_k)\frac{\hat{w}_k}{1 - x_k^2}, \quad k = 2, \dots, n-1.$$

- $m = 3$:

$$w_{k,1} = -\frac{3\omega_n''(x_k)}{2\omega_n'(x_k)}w_{k,0}, \quad w_{k,2} = \left(-\frac{\omega_n^{(3)}(x_k)}{2\omega_n'(x_k)} + \frac{3}{2}\left(\frac{\omega_n''(x_k)}{\omega_n'(x_k)}\right)^2\right)w_{k,0}.$$

- $m = 4$:

$$w_{k,1} = -\frac{2\omega_n''(x_k)}{\omega_n'(x_k)}w_{k,0}, \quad w_{k,2} = \left(-\frac{2\omega_n^{(3)}(x_k)}{3\omega_n'(x_k)} + \frac{5}{2}\left(\frac{\omega_n''(x_k)}{\omega_n'(x_k)}\right)^2\right)w_{k,0}$$

and

$$w_{k,3} = \left(-\frac{\omega_n^{(4)}(x_k)}{6\omega_n'(x_k)} + \frac{5\omega_n''(x_k)\omega_n^{(3)}(x_k)}{3\omega_n'(x_k)\omega_n'(x_k)} - \frac{5}{2}\left(\frac{\omega_n''(x_k)}{\omega_n'(x_k)}\right)^3\right)w_{k,0}.$$

3. Illustration of numerical stability and numerical examples. The stability for the second barycentric formulas for Lagrange interpolation has been extensively studied by Henrici [27], Berrut and Trefethen [5]. Rigorous arguments that make this intuitive idea precise are provided by Higham [28, 29]. For more details, see [5, 28, 29].

These arguments can be directly applied to the second barycentric higher order Hermite-Fejér formula (2.15) if the barycentric weights can be evaluated well and do not suffer from overflows or underflows, which makes barycentric interpolation entirely reliable in practice for small values of m at the pointsystems, such as the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem for $\alpha = \beta = 1.5$, i.e. the roots of $(1 - x^2)P_{n-2}^{(\frac{3}{2}, \frac{3}{2})}(x)$, studied in this paper.

To be pointed out especially, although both the **Algorithm 1** and **Algorithm 2** enjoy the fast implementation at the cost of $O(nm^2)$ operations for pointsystems discussed in Section 2, the performances are not always same. More specifically, they own the same high accuracy for small m and different outcomes for larger m . The **Algorithm 2** manifests better stability for larger m in our numerical experiments. Form the descriptions of the two algorithms, we can see that the first algorithm needs one more step than the second algorithm. This extra step leads to a great loss of significance due to the fast growth of the entries in these two algorithms for large m . So, the **Algorithm 2** is recommended in practice.

Here, we use four functions to test the accuracy and stability for the second Hermite-Fejér barycentric interpolation form with **Algorithm 2** at the Gauss-Jacobi and Jacobi-Gauss-Lobatto pointsystems with $f(x) = 1/(1+x^2)$, which is analytic in a neighborhood of $[-1, 1]$, C^∞ function $f(x) = e^{-1/x^2}$, and nonsmooth functions $f(x) = 1 - |x|^3$. Figures 3.1-3.3 illustrate the performance of the second barycentric interpolant (2.15) for the above first three functions at Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$ by using $n = 10 : 20 : 2000$ grid points and $m = 2^{j-1}$ ($j = 1, 2, \dots, 6$) derivatives under the ∞ -norm for the vector $\|f(x) - H_{mn-1}(f, x)\|_\infty$ at $x = -1 : 0.02 : 1$.

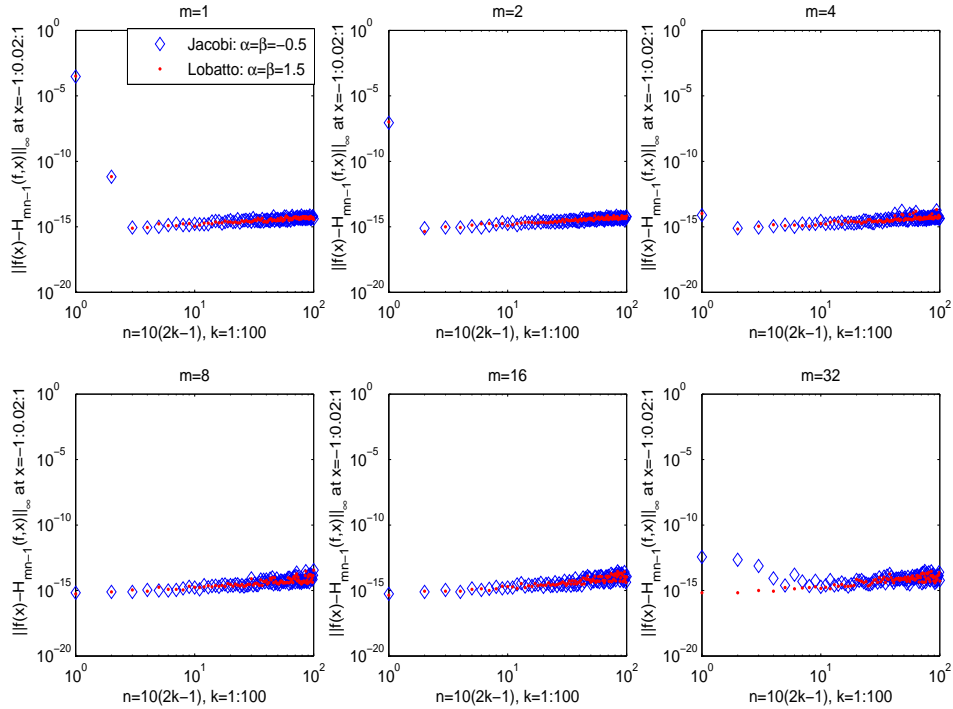


FIG. 3.1. $\|f(x) - H_{mn-1}(f, x)\|_\infty$ at $x = -1 : 0.02 : 1$ with $n = 10 : 20 : 2000$ and $m = 2^{j-1}$ ($j = 1, 2, \dots, 6$) for $f(x) = 1/(1+x^2)$ at the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$, respectively.

From these examples, we can see that the barycentric interpolation is quite stable for small values of m . However, when m is too large, the simplified barycentric weights will suffer from overflows or underflows too. Figure 3.5 shows the maximum m for a fixed n in the computation

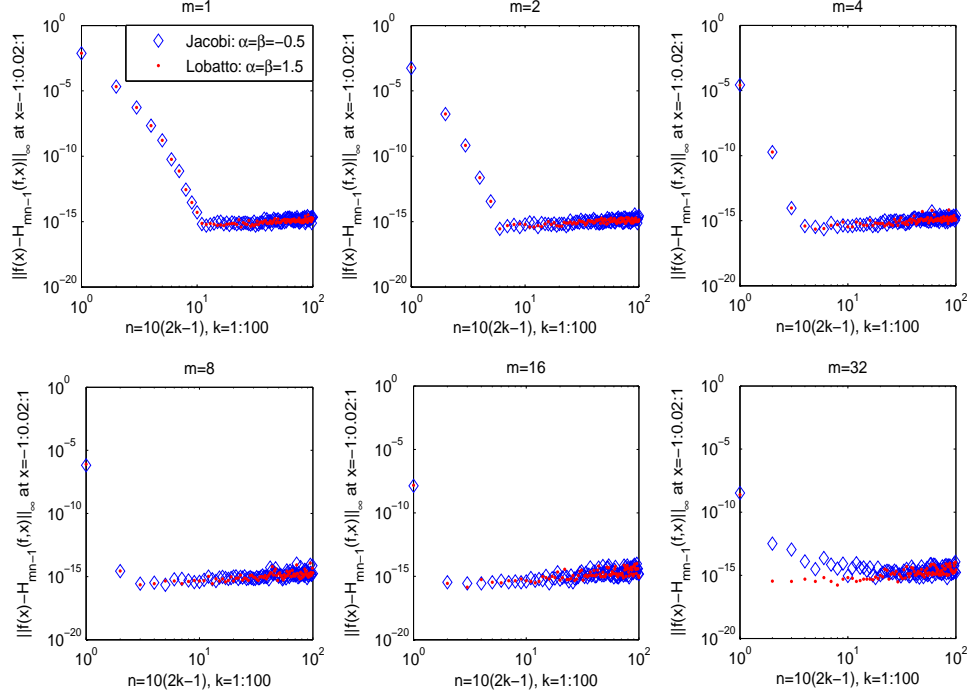


FIG. 3.2. $\|f(x) - H_{mn-1}(f, x)\|_{\infty}$ at $x = -1 : 0.02 : 1$ with $n = 10 : 20 : 2000$ and $m = 2^{j-1}$ ($j = 1, 2, \dots, 6$) for $f(x) = e^{-1/x^2}$ at the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$, respectively.

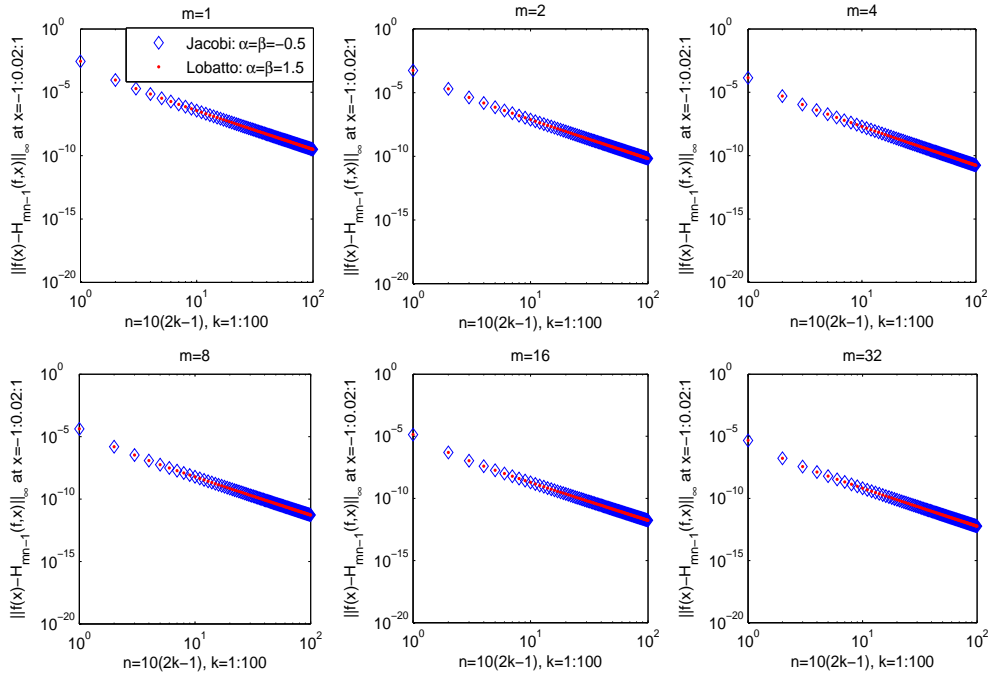


FIG. 3.3. $\|f(x) - H_{mn-1}(f, x)\|_{\infty}$ at $x = -1 : 0.02 : 1$ with $n = 10 : 20 : 2000$ and $m = 2^{j-1}$ ($j = 1, 2, \dots, 6$) for $f(x) = 1 - |x|^3$ at the Chebyshev pointsystem (1.3) and Jacobi-Gauss-Lobatto pointsystem with $\alpha = \beta = 1.5$, respectively.

of the simplified barycentric weights $w_{k,r}$ ($k = 1 : n, r = 0 : m - 1$) by **Algorithm 2** before the overflows or underflows occur.

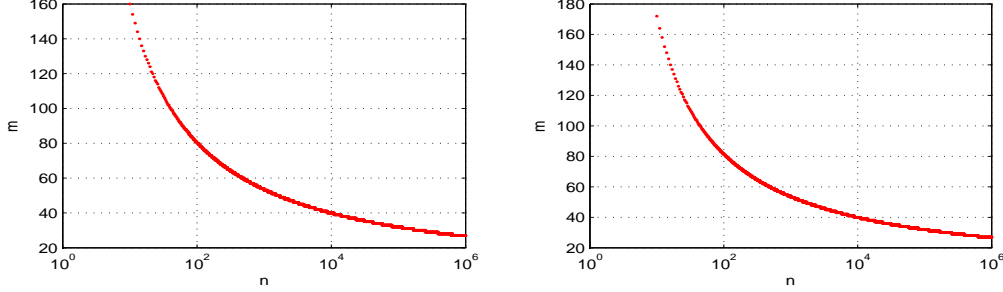


FIG. 3.4. The maximum number m for a fixed n before the overflows or underflows occurred for Gauss-Jacobi pointsystem $\alpha = \beta = -0.5$ (left) and for Jacobi-Gauss-Lobatto pointsystem $\alpha = \beta = 1.5$ (right): $n = 10 : 10^6$.

4. Final remarks. It is remarkable that Chebyshev pointsystems (1.3) and (2.32) are fairly nice in Lagrange polynomial approximation (see [49, 50, 59]). However, for (higher order) Hermite-Fejér interpolation, the Chebyshev pointsystem (2.32) completely fails (see Figure 4.1). The good choice is Chebyshev pointsystem (1.3) or the roots of $(1 - x^2)P_{n-2}^{(\frac{3}{2}, \frac{3}{2})}(x)$, since pointsystem (2.32) is not normal and the latter two pointsystems are strongly normal for $m = 2$.

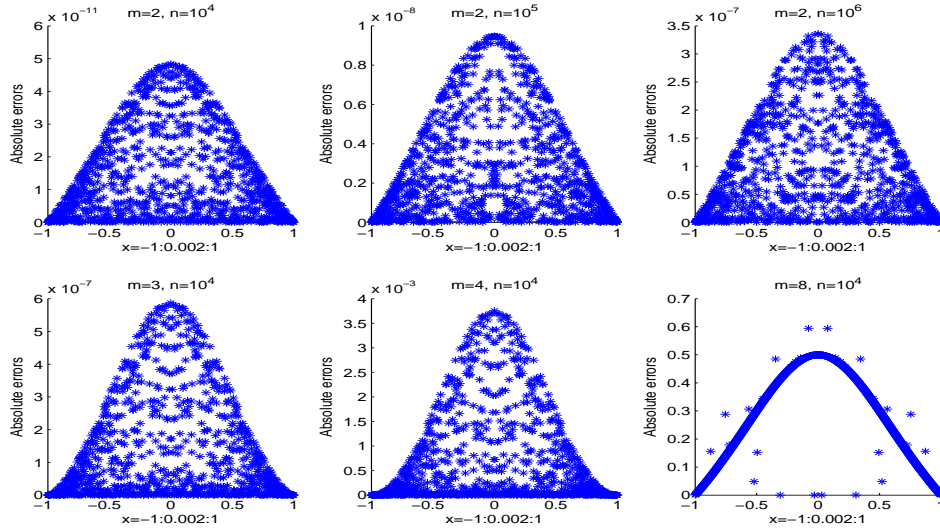


FIG. 4.1. The absolute errors of $H_{mn-1}(f, x) - f(x)$ at $x = -1 : 0.002 : 1$ with different m and n by using Chebyshev pointsystem (2.32) for $f(x) = \frac{1}{1+x^2}$.

For strongly normal pointsystem satisfying (1.6), Vértesi [52] proved that for each $f \in C^1[-1, 1]$,

$$\|E(f)\|_\infty = \max_{x \in [-1, 1]} |H_{2n-1}(f, x) - f(x)| \leq \left(4 + \frac{2}{c}\right) \min_{q_j \in \mathcal{P}_{2n-2}} \|f' - q_j\|_\infty$$

where \mathcal{P}_{2n-2} denotes the set of all polynomials of degree at most $2n - 1$ with real coefficients.

If f is analytic or of finite limited regularity, the convergence rate on Hermite-Fejér interpolation $H_{2n-1}(f, x)$ at Gauss-Jacobi pointsystem can be improved and given explicitly based on the asymptotics of the coefficients of Chebyshev series for f .

Suppose $f(x)$ satisfies a Dini-Lipschitz condition on $[-1, 1]$, then it has the following absolutely and uniformly convergent Chebyshev series expansion (see Cheney [11, pp 129])

$$(4.1) \quad f(x) = \sum_{j=0}^{\infty} 'c_j T_j(x), \quad c_j = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_j(x)}{\sqrt{1-x^2}} dx, \quad j = 0, 1, \dots$$

where the prime denotes summation whose first term is halved, $T_j(x) = \cos(j \cos^{-1} x)$ denotes the Chebyshev polynomial of degree j .

LEMMA 4.1. (i) (Bernstein [4]) If f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse \mathcal{E}_ρ with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $j \geq 0$,

$$(4.2) \quad |c_j| \leq \frac{2M}{\rho^j}.$$

(ii) (Trefethen [49, 50]) For an integer $k \geq 1$, if $f(x)$ has an absolutely continuous $(k-1)$ st derivative $f^{(k-1)}$ on $[-1, 1]$ and a k th derivative $f^{(k)}$ of bounded variation $V_k = \text{Var}(f^{(k)}) < \infty$, then for each $j \geq k+1$,

$$(4.3) \quad |c_j| \leq \frac{2V_k}{\pi j(j-1) \cdots (j-k)}.$$

LEMMA 4.2. Suppose $\{x_j\}_{j=1}^n$ are the roots of $P_n^{(\alpha, \beta)}(x)$ ($\alpha, \beta > -1$), then it follows

$$(4.4) \quad (x - x_j) \ell_j(x) = \sigma_n \frac{\sqrt{(1-x_j^2)\overline{w}_j}}{2^{(\alpha+\beta+1)/2}} \sqrt{\frac{n! \Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}} P_n^{(\alpha, \beta)}(x), \quad j = 1, 2, \dots, n.$$

Proof. Let $z_n = \int_{-1}^1 (1-x)^\alpha (1+x)^\beta [P_n^{(\alpha, \beta)}(x)]^2 dx$ and K_n the leading coefficient of $P_n^{(\alpha, \beta)}(x)$. From Abramowitz and Stegun [1], we have

$$z_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{n! \Gamma(n+\alpha+\beta+1)}, \quad K_n = \frac{1}{2^n} \frac{\Gamma(2n+\alpha+\beta+1)}{n! \Gamma(n+\alpha+\beta+1)}.$$

Furthermore, by Hale and Townsend [25] and Wang et al. [57], we obtain

$$\begin{aligned} (x - x_j) \ell_j(x) = \frac{w_n(x)}{w'_n(x_j)} &= \sigma_n (-1)^j \sqrt{\frac{K_n^2 2n(1-x_j^2)\overline{w}_j}{2n(2n+\alpha+\beta+1)z_n}} w_n(x) \\ &= \sigma_n (-1)^j \sqrt{\frac{(1-x_j^2)\overline{w}_j}{(2n+\alpha+\beta+1)z_n}} P_n^{(\alpha, \beta)}(x), \end{aligned}$$

which leads to the desired result (4.4). \square

THEOREM 4.3. Suppose $\{x_j\}_{j=1}^n$ are the roots of $P_n^{(\alpha, \beta)}(x)$ ($-1 < \alpha, \beta \leq 0$), then the Hermite-Fejér interpolation (1.5) at $\{x_j\}_{j=1}^n$ has the convergence rate

$$(4.5) \quad \|E(f)\|_\infty \leq \begin{cases} \frac{4\tau_n M [2n\rho^2 + (1-2n)\rho]}{(\rho-1)^2 \rho^{2n}} & (n \geq 1), \quad \text{if } f \text{ analytic in } \mathcal{E}_\rho \text{ with } |f(z)| \leq M \\ \frac{4\tau_n V_k}{(k-1)\pi(2n-1)(2n-2) \cdots (2n-k+1)}, & \text{if } f, \dots, f^{(k-1)} \text{ absolutely continuous} \\ & \text{and } V_k < \infty, n \geq k/2, k \geq 2, \end{cases}$$

where $E(f, x) = f(x) - H_{2n-1}(f, x)$, and

$$(4.6) \quad \tau_n = \begin{cases} O(n^{-1.5-\min\{\alpha, \beta\}} \log n), & \text{if } -1 < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq -\frac{1}{2} \\ O(n^{2\max\{\alpha, \beta\} - \min\{\alpha, \beta\} - \frac{1}{2}}), & \text{if } -1 < \min\{\alpha, \beta\} \leq -\frac{1}{2} < \max\{\alpha, \beta\} \leq 0 \\ O(n^{2\max\{\alpha, \beta\}}), & \text{if } -\frac{1}{2} < \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq 0 \end{cases}.$$

Proof. Since the Chebyshev series expansion of $f(x)$ is uniformly convergent under the assumptions of Theorem 4.3, and the error of Hermite-Fejér interpolation (1.5) on Chebyshev polynomials satisfies $|E(T_j, x)| = |T_j(x) - H_{2n-1}(T_j, x)| = 0$ for $j = 0, 1, \dots, 2n-1$, then it yields

$$(4.7) \quad |E(f, x)| = |f(x) - H_{2n-1}(f, x)| = \left| \sum_{j=0}^{\infty} |c_j| |E(T_j, x)| \right| \leq \sum_{j=2n}^{\infty} |c_j| |E(T_j, x)|.$$

Furthermore, $|E(T_j, x)| = |T_j(x) - \sum_{i=1}^n T_j(x_i) h_i(x) - \sum_{i=1}^n T'_j(x_i) b_i(x)|$. In the following, we will focus on estimates on $|E(T_j, x)|$ for $j \geq 2n$.

Notice that the pointsystem is normal which implies $h_i(x) \geq 0$ for all $i = 1, 2, \dots, n$ and $x \in [-1, 1]$,

$$1 \equiv \sum_{i=1}^n h_i(x) = \sum_{i=1}^n v_i(x) \ell_i^2(x)$$

(see [16]) and then

$$(4.8) \quad \left| \sum_{i=1}^n T_j(x_i) h_i(x) \right| \leq \sum_{i=1}^n h_i(x) = 1, \quad j = 0, 1, \dots$$

Additionally, by Lemma 4.2, it obtains for $j = 2n, 2n+1, \dots$ that

$$\begin{aligned} & \left| \sum_{i=1}^n T'_j(x_i) b_i(x) \right| \\ &= j \left| \sum_{i=1}^n U_{j-1}(x_i) (x - x_i) \ell_i^2(x) \right| \\ &= \frac{j}{2^{(\alpha+\beta+1)/2}} \sqrt{\frac{n! \Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}} |P_n^{(\alpha, \beta)}(x) \sum_{i=1}^n U_{j-1}(x_i) \sqrt{(1-x_i^2) \bar{w}_i} \ell_i(x)| \\ &= \frac{j}{2^{(\alpha+\beta+1)/2}} \sqrt{\frac{n! \Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}} |P_n^{(\alpha, \beta)}(x) \sum_{i=1}^n \sin((j-1) \arccos(x_i)) \sqrt{\bar{w}_i} \ell_i(x)| \\ &= j O \left(|P_n^{(\alpha, \beta)}(x)| \sqrt{\|\{\bar{w}_i\}_{i=1}^n\|_{\infty} \Lambda_n} \right) \end{aligned}$$

since $\sqrt{\frac{n! \Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}}$ is decreasing as n increases and then uniformly bounded on n for $-1 < \alpha, \beta \leq 0$, where $\Lambda_n = \max_{x \in [-1, 1]} \sum_{i=1}^n |\ell_i(x)|$ is the Lebesgue constant, which, together with

$$P_n^{(\alpha, \beta)}(x) = \begin{cases} O(n^{-\frac{1}{2}}), & \text{if } \max\{\alpha, \beta\} \leq -\frac{1}{2} \\ O(n^{\max\{\alpha, \beta\}}), & \text{if } \max\{\alpha, \beta\} > -\frac{1}{2} \end{cases}, \quad \bar{w}_i = \begin{cases} O(n^{-2-2\min\{\alpha, \beta\}}), & \text{if } \min\{\alpha, \beta\} \leq -\frac{1}{2} \\ O(n^{-1}), & \text{if } \min\{\alpha, \beta\} > -\frac{1}{2} \end{cases}$$

(see Szegő [45, pp 168, 354]) and

$$\Lambda_n = \begin{cases} O(\log n), & \text{if } \max\{\alpha, \beta\} \leq -\frac{1}{2} \\ O(n^{\max\{\alpha, \beta\} + \frac{1}{2}}), & \text{if } \max\{\alpha, \beta\} > -\frac{1}{2} \end{cases} \quad ([45, \text{pp } 338]),$$

yields

$$(4.9) \quad \left| \sum_{i=1}^n T'_j(x_i) b_i(x) \right| = j \tau_n.$$

Thus, by (4.8), (4.9) and (1.5), we find $|E(T_j, x)| \leq 2 + j \tau_n$ for $j \geq 2n$, and then the error of Hermite-Fejér interpolation (4.7) satisfies

$$|E(f, x)| = |f(x) - H_{2n-1}(f, x)| \leq \sum_{j=2n}^{\infty} |c_j| |E(T_j, x)| = 2 \tau_n \sum_{j=2n}^{\infty} j |c_j|,$$

which, following [59], leads to the desired result. \square

From the definition of τ_n (4.6), we see that when $\alpha = \beta = -\frac{1}{2}$ the convergence order on n is the lowest. In addition, from Szabados [44] (also see Sadiq and Viswanath [36]), we see that the convergence of the higher order Hermite-Fejér interpolation (2.15) at the Chebyshev pointsystem (1.3) satisfies

$$(4.10) \quad \|f - H_{mn-1}(f)\|_\infty = \begin{cases} O(\log n) \|f - p^*\|_{C^{m-1}[-1,1]}, & \text{if } m \text{ is odd} \\ O(1) \|f - p^*\|_{C^{m-1}[-1,1]}, & \text{if } m \text{ is even} \end{cases}$$

where p^* is the best approximation polynomial of f with degree at most $mn - 1$ and $\|f - p^*\|_{C^{m-1}[-1,1]} = \max_{0 \leq j \leq m-1} \|f^{(j)} - (p^*)^{(j)}\|_\infty$.

Numerical examples also illustrate that the roots of $(1 - x^2)P_{n-2}^{(\frac{3}{2}, \frac{3}{2})}(x)$ are appropriate to higher order Hermite-Fejér interpolation. In the future work, we will consider the convergence rates on this pointsystem.

It is worth noting that the new methods for Hermite barycentric weights at Gauss-Jacobi pointsystems or Jacobi-Gauss-Lobatto pointsystems can be extended to Jacobi-Gauss-Radau pointsystems or the roots of other kinds of orthogonal polynomials, such as Laguerre polynomials, Hermite polynomials, etc., based on the works of [20], [57] and CHEBFUN [51].

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